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Bayesian Estimation of Stress-Strength Reliability Function $P(T<X<Z)$ for EIR Distribution

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Abstract—

This study presents the derivation of the stress-strength reliability (S-SR) function P(T<X<Z), assuming complete data. In this case, the component strength X lies between two independent stresses T and Z, with all three variables independently following the Exponentiated Inverse Rayleigh (EIR) distribution. The scale parameter is assumed to be common and known, while the shape parameters differ. The reliability function is estimated using the Bayesian approach under both informative gamma priors and non-informative priors, based on the Quadratic Loss Function (QLF). These Bayesian estimators are compared to the Maximum Likelihood Estimator (MLE). A simulation study is conducted to evaluate and compare the estimators' performance. The findings reveal that the Bayesian estimator based on the informative gamma prior under the QLF provides the most accurate and efficient results.

Keywords*— Stress- strength reliability, Bayesian estimation, P(T<X<Z), Quadratic loss function, Exponentiated inverse Rayleigh distribution.*

I. Introduction

The Exponentiated Inverse Rayleigh Distribution (EIRD) is a flexible lifetime distribution that has gained attention in the fields of reliability analysis and statistical quality control. The EIRD is particularly useful for modeling failure times and has applications in engineering. It is a generalization from the standard Inverse Rayleigh Distribution by introducing an additional shape parameter, which enhances its ability to model a wider range of data behaviors, particularly those with monotonic or non-monotonic hazard rates. This distribution was proposed by Nadarajah and Kotz [1], who introduced a systematic method for generating new distributions by exponentiating the reliability function:

$$F(x) = 1 - \{R(x)\}^{\alpha} \qquad (1)$$

Where R(x) is reliability function for Inverse Rayleigh distribution.

Using equation (1)

The CDF of Exponentiated Inverse Rayleigh distribution is $F(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha}, \ \alpha > 0, \qquad (2)$

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And the pdf of EIR distribution is:

$$f(x) = \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1}, \quad x > 0\,, \alpha > 0, \quad \lambda > 0, \qquad (3)$$

Where α is the shape parameter and λ is the scale parameter.

When $\alpha = 1$ the (EIR) dist. returns back to Inverse Rayleigh (IR) distribution.

The stress-strength reliability (S-SR) have two main types: classical and modern. The classical stress-strength model focuses on evaluating the lifetime of a component by assessing its ability (strength X) to withstand random stress T. The primary interest lies in estimating the probability that the component's strength exceeds the applied stress, expressed as P(T<X), The component may fail, or the system in which it operates might malfunction, when the applied stress exceeds the component's strength $(T \geq X)$.

The second type is P(T<X<Z); study and focuses on evaluating and estimating the probability P(T<X<Z), which reflects a more nuanced reliability scenario where the component's strength X must not only exceed a lower stress threshold T, but also remain below an upper stress limit Z. This form of stress-strength reliability is applicable in situations where the operational range is bounded by two limits. A practical example is blood pressure, where a healthy reading must fall between two values—systolic and diastolic pressures—indicating that the measurement should be neither too low nor too high, [Kotz [2]]. Over the past 45 years, Singh [3] examined a case in which the cumulative distribution functions (CDFs) of X and Z are known, while the probability density function (PDF) of Y is unknown but observed data from Y are available. Amal, Elsayed and Rania [4] The reliability was estimated under assumption that X, Y, and Z are independent and follow the Weibull distribution, sharing a common known shape parameter but having different scale parameters; The analysis also considered the presence of k outliers in the strength variable X. Moment estimators, Maximum Likelihood Estimators (MLE), and mixture estimators were derived for this case. Patowary, Sriwastav and Hazarika [5] demonstrated the estimation of reliability R=P(X<Y<Z) using Monte Carlo Simulation (MCS) in an *n*-standby system, where both stress and strength variables follow a continuous distribution. Ali and Nada [6] estimated the reliability using various methods including Maximum Likelihood, Method of Moments, Least Squares, and Weighted Least Squares; assuming that X, T, and Z follow the New Weibull-Pareto Distribution with a combination of known and unknown parameters.

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II. S-SR Reliability formula

Deriving The formula of the stress-strength reliability function P(T<X<Z) for complete data when component's strength (X) falls in between the stresses T and Z respectively, will be as follows [7] :

$$R = P(T < X < Z) = \int_0^\infty P(T < X, X < Z) f(x) dx,$$

where X, T, Z are independent

$$= \int_0^\infty H_t(x)\,\overline{G_Z}(x)\, f(x)\, dx,$$

$$= \int_0^\infty H_t(x)\left(1 - G_z(x)\right) f(x) dx,$$

$$R = \int_0^\infty H_t(x) f(x)\, dx - \int_0^\infty H_t(x)\, G_z(x)\, f(x) dx,$$

Suppose that

$$A_1 = \int_0^\infty H_t(x) f(x)\, dx.$$

$$A_2 = \int_0^\infty H_t(x)\, G_z(x)\, f(x) dx.$$

$$R = A_1 - A_2, \qquad (4)$$

T and Z independent random stress variables following the Exponentiated Inverse Rayleigh (EIR) distributions with parameters$EIR(\alpha_1, \lambda)$, $EIR(\alpha_2, \lambda)$ Respectively,whereα_1, α_2 are the shape parameters and λ the scale parameter, and (X) is random strength and independent from T and Z and also follow EIR (α, λ) then :

$$H_t(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1}, \quad x > 0, \lambda;\ \alpha_1 > 0.$$

$$G_z(x) = 1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2}, \quad x > 0, \lambda;\ \alpha_1 > 0.$$

$$A_1 = \int_0^\infty (1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1}) f(x) dx,$$

$$= \int_0^\infty f(x) dx - \int_0^\infty \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} f(x) dx,$$

$$= 1 - \int_0^\infty \frac{2\,\alpha\,\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1} * \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} dx,$$

By solving this equation

$$= 1 - \frac{\alpha}{\alpha_1 + \alpha} \int_0^\infty \frac{2(\alpha_1 + \alpha)\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} * \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{(\alpha+\alpha_1)-1} dx,$$

$$A_1 = 1 - \frac{\alpha}{\alpha+\alpha_1}(1) = \frac{\alpha_1}{\alpha_1+\alpha}. \qquad (5)$$

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$$A_2 = \int_0^\infty H_t(x)\, G_z(x) f(x) dx,$$

$$A_2 = \int_0^\infty \left[1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1}\right] * \left[1 - \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2}\right] \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1} dx,$$

Multiply the parentheses and insert the integral

$$A_2 = 1 - \int_0^\infty \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1} * \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} dx$$
$$- \int_0^\infty \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2} dx$$
$$+ \int_0^\infty \frac{2\alpha\lambda^2}{x^3} e^{-\left(\frac{\lambda}{x}\right)^2} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha-1} \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_1} * \left(1 - e^{-\left(\frac{\lambda}{x}\right)^2}\right)^{\alpha_2} dx,$$

$$A_2 = 1 - \frac{\alpha}{\alpha+\alpha_1} - \frac{\alpha}{\alpha+\alpha_2} + \frac{\alpha}{\alpha+\alpha_1+\alpha_2}. \qquad (6)$$

Substituting equation (5) and equation (6) in equation (4)

$$R = \frac{\alpha\alpha_1}{(\alpha+\alpha_2)(\alpha+\alpha_1+\alpha_2)}. \qquad (7)$$

III. Maximum likelihood estimation

A. *Likelihood Function*

Let $\{x_i\}$be the random sample of strength from size (n), (i = 1,2,…,n) from EIR (α, λ) With known scale parameter $\lambda > 0$ and unknown shape parameter $\alpha > 0$, then the likelihood function of $\underline{x}$ is:

$$L\left(\underline{x}|\alpha\right) = \prod_{i=1}^{m} f(x|\alpha), \qquad i = 1,2,\dots,n.$$

$$L\left(\underline{x}|\alpha\right) = 2^n \alpha^n \lambda^{2n} \left(\prod_{i=1}^{n} x_i^{-3}\right) e^{-\sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^2} * \prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha-1},$$

$$= 2^n \alpha^n \lambda^{2n} \left(\prod_{i=1}^{n} x_i^{-3}\right) e^{-\sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^2}$$

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$$* \prod_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha}\prod_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-1},$$

$$L\left(\underline{x}|\alpha\right)=B\,\alpha^{n}\prod_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha}. \qquad (8)$$

Where:

$$B=2^{n}\lambda^{2n}\left(\prod_{i=1}^{n}x_i^{-3}\right)e^{-\sum_{i=1}^{n}\left(\frac{\lambda}{x_i}\right)^2}\prod_{i=1}^{n}\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{-1}.$$

And let t_j *and* z_k be the random samples represent stress from sizes (m , w) , (j = 1,2,…,m ; k = 1,2,…,w) from $EIR(\alpha_1,\lambda)$, $EIR(\alpha_2,\lambda)$ Respectively with shape parameters $\alpha_1,\alpha_2>0$ are unknown and the scale parameter $\lambda>0$ is known , t_j *and* z_k are independent from each other and from x_i, then the likelihood functions of $\underline{t}$, $\underline{z}$ are :

$$L\left(\underline{t}|\alpha_1\right)=B_1\,\alpha_1^{m}\prod_{j=1}^{m}\left(1-e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)^{\alpha_1}. \qquad (9)$$

Where:

$$B_1=2^{m}\lambda^{2m}\left(\prod_{j=1}^{m}t_j^{-3}\right)e^{-\sum_{j=1}^{m}\left(\frac{\lambda}{t_j}\right)^2}\prod_{j=1}^{m}\left(1-e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)^{-1}.$$

$$L\left(\underline{z}|\alpha_2\right)=B_2\,\alpha_2^{w}\prod_{k=1}^{w}\left(1-e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)^{\alpha_2}. \qquad (10)$$

Where:

$$B_2=2^{w}\lambda^{2w}\left(\prod_{k=1}^{w}z_k^{-3}\right)e^{-\sum_{k=1}^{w}\left(\frac{\lambda}{z_k}\right)^2}\prod_{k=1}^{w}\left(1-e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)^{-1}.$$

B. *Maximum likelihood estimator (M.L.E):*

the shape parameter (α) can be estimated as follows:

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$\ln L\,(\underline{x}|\alpha) = \ln B + n\ln\alpha + \alpha\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)$, $\frac{\partial \ln(\underline{x}|\alpha)}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)$,

$$\Rightarrow \frac{n}{\hat{\alpha}} + \sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right) = 0\,, \qquad (11)$$

Then by solving equation (11):

$$\hat{\alpha}(mle) = \frac{-n}{\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)}. \qquad (12)$$

Similarly, the M.L.E.'S for α_1, α_2 are:

$$\hat{\alpha}_1(mle) = \frac{-m}{\sum_{j=1}^{m}\ln\left(1-e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)}. \qquad (13)$$

$$\hat{\alpha}_2(mle) = \frac{-w}{\sum_{k=1}^{w}\ln\left(1-e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)}. \qquad (14)$$

The Maximum Likelihood Estimator (MLE) of the stress-strength reliability can be obtained by utilizing the invariance property of maximum likelihood estimation of $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2$ that in equations (12), (13), (14) as follows:

$$\hat{R}(mle) = \frac{\hat{\alpha}(mle)\hat{\alpha}_1(mle)}{(\hat{\alpha}(mle)+\hat{\alpha}_2(mle))(\hat{\alpha}(mle)+\hat{\alpha}_1(mle)+\hat{\alpha}_2(mle))}. \qquad (15)$$

IV. Bayesian estimation

This section focuses on estimating the stress-strength reliability using the Bayesian estimation methods, assuming complete data. The estimation is carried out using both informative and non-informative priors, based on the Quadratic Loss Function (QLF):

Bayesian estimation based on Non-informative Jeffrey's prior :

The non-informative Jeffrey's prior for the parameter α is define as follow [8]:

$$P(\alpha) \propto \sqrt{I_X(\alpha)},$$

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$I_X(\alpha)$ is the fisher information for α which take the following formula

$$I_X(\alpha) = -E\left[\frac{\partial^2 \ln f(\underline{x}|\alpha)}{\partial\alpha^2}\right] = \frac{1}{\alpha^2},$$

The non-informative prior for α is

$$P(\alpha) \propto \frac{1}{\alpha}, \qquad (16)$$

The posterior distribution for α is

$$P(\alpha|\underline{x})_J \propto L(\underline{x}|\alpha)\, P(\alpha),$$

Substituting equations (8) and (16)

$$P(\alpha|\underline{x})_J = \alpha^{n-1} e^{\alpha\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)},$$

Which is the kernel of gamma distribution G (n , D)

Where D = $-\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)$,

Then The complete posterior dist. Of $P(\alpha|\underline{x})_J$ is:

$$P(\alpha|\underline{x})_J = \frac{D^n}{\Gamma n}\alpha^{n-1}e^{-\alpha D}. \qquad (17)$$

Similarly, the posterior distribution for $\alpha_1 and\ \alpha_2$ are

$$P(\alpha_1|\underline{t})_J = \frac{D^m}{\Gamma m}\alpha_1{}^{m-1}e^{-\alpha_1 D_1}. \qquad (18)$$

Where $D_1 = -\sum_{j=1}^{m}\ln\left(1-e^{-\left(\frac{\lambda}{t_j}\right)^2}\right)$,

$$P(\alpha_2|\underline{z})_J = \frac{D_2^w}{\Gamma w}\alpha_2^{w-1}e^{-\alpha_2 D_2}. \qquad (19)$$

Where $D_2 = -\sum_{k=1}^{w}\ln\left(1-e^{-\left(\frac{\lambda}{z_k}\right)^2}\right)$,

Since $\underline{x}, \underline{t}$ *and* $\underline{z}$ *are independent* The joint posterior can be found by multiplying equations (17) (18) (19):

$$P(\alpha,\alpha_1,\alpha_2|\underline{x},\underline{t},\underline{z})_J = \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w}\alpha^{n-1}\alpha_1{}^{m-1}\alpha_2^{w-1}e^{-\alpha D}e^{-\alpha_1 D_1}e^{-\alpha_2 D_2}. \qquad (20)$$

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Bayesian estimation using Non informative Jeffrey's prior based on Quadratic loss function (Q.L.F):

The quadratic loss function takes the following form [9]:

$$L(R,\hat{R}) = \left(\frac{\hat{R}-R}{R}\right)^2,$$

To find the Bayesian estimation ($\hat{R}$) for(S-SR.) based on quadratic loss function (Q.L.F) we solved the following equation:

$$\frac{\partial E[L(R,\hat{R})]}{\partial \hat{R}} = 0,$$

$$\hat{R} = \frac{E\left(R^{-1}|\underline{x},\underline{t},\underline{z}\right)}{E\left(R^{-2}|\underline{x},\underline{t},\underline{z}\right)}, \quad \text{Then} \quad \hat{R}_J = \frac{E\left(R^{-1}|\underline{x},\underline{t},\underline{z}\right)_J}{E\left(R^{-2}|\underline{x},\underline{t},\underline{z}\right)_J}, \qquad (21)$$

The expectation in the numerator of equation (21) using equation (7):

$$E\left(R^{-1}|\underline{x},\underline{t},\underline{z}\right)_J = \int_0^\infty\int_0^\infty\int_0^\infty R^{-1}P\left(\alpha,\alpha_1,\alpha_2|\underline{x},\underline{t},\underline{z}\right)_J d\alpha\, d\alpha_1 d\alpha_2\,,$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty (1+2\alpha_1^{-1}\alpha_2+\alpha^{-1}\alpha_2+\alpha^{-1}\alpha_1^{-1}\alpha_2^2+\alpha\alpha_1^{-1}) * \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w}\alpha^{n-1}\alpha_1{}^{m-1}\alpha_2^{w-1}e^{-\alpha D}e^{-\alpha_1 D_1}e^{-\alpha_2 D_2}\, d\alpha\, d\alpha_1 d\alpha_2\,,$$

$$E\left(R^{-1}|\underline{x},\underline{t},\underline{z}\right)_J = \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w} * \Big[\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n-1}\,\alpha_1{}^{m-1}\alpha_2{}^{w-1} * e^{-D\alpha}\; e^{-D_1\alpha_1}\, e^{-D\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2 + 2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n-1}\,\alpha_1{}^{(m-1)-1}\alpha_2{}^{(w+1)-1} * e^{-D\alpha}\; e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-1)-1}\,\alpha_1{}^{m-1}\alpha_2{}^{(w+1)-1} * e^{-D\alpha}\; e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-1)-1}\,\alpha_1{}^{(m-1)-1}\alpha_2{}^{(w+2)-1} * e^{-D\alpha}\; e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+1)-1}\,\alpha_1{}^{(m-1)-1}\alpha_2{}^{w-1} * e^{-D\alpha}\; e^{-D_1\alpha_1}\, e^{-D_2\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2\Big],$$

By solving the integrations which is kernels of gamma dist.

$$= \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w}\left[\frac{\Gamma(n)}{D^n}\frac{\Gamma(\text{m})}{D_1{}^m}\frac{\Gamma(\text{w})}{D_2{}^w} + 2\,\frac{\Gamma(n)}{D^n}\frac{\Gamma(\text{m}-1)}{D_1{}^{m-1}}\frac{\Gamma(\text{w}+1)}{D_2{}^{w+1}} + \frac{\Gamma(n-1)}{D^{n-1}}\frac{\Gamma(\text{m})}{D_1{}^m}\frac{\Gamma(\text{w}+1)}{D_2{}^{w+1}} + \frac{\Gamma(n-1)}{D^{n-1}}\frac{\Gamma(\text{m}-1)}{D_1{}^{m-1}}\frac{\Gamma(\text{w}+2)}{D_2{}^{w+2}} + \frac{\Gamma(n+1)}{D^{n+1}}\frac{\Gamma(\text{m}-1)}{D_1{}^{m-1}}\frac{\Gamma(\text{w})}{D_2{}^w}\right],$$

$$E\left(R^{-1}|\underline{x},\underline{t},\underline{z}\right)_J = 1 + 2\frac{(\text{w})D_1}{(\text{m}-1)D_2} + \frac{(\text{w})D}{(n-1)D_2} + \frac{(\text{w}+1)(\text{w})DD_1}{(n-1)(\text{m}-1)D_2^2} + \frac{(\text{n})D_1}{(\text{m}-1)D}. \qquad (22)$$

The expectation in the denominator of equation (21) using equation (7):

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$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right)_J = \int_0^\infty\int_0^\infty\int_0^\infty R^{-2}P\left(\alpha,\alpha_1,\alpha_2\middle|\underline{x},\underline{t},\underline{z}\right)_J d\alpha\, d\alpha_1 d\alpha_2\,,$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty (6\alpha_1^{-2}\,\alpha_2^2 + \alpha^{-2}\alpha_2^2 + \alpha^{-2}\alpha_1^{-2}\alpha_2^4 + 6\alpha^{-1}\alpha_1^{-1}\alpha_2^2 + 4\alpha^{-1}\alpha_1^{-2}\alpha_2^3 + 2\alpha^{-2}\alpha_1^{-1}\alpha_2^3 + 4\alpha\alpha_1^{-2}\alpha_2 + 2\alpha^{-1}\alpha_2 + 6\alpha_1^{-1}\alpha_2 + \alpha^2\alpha_1^{-2} + 1 + 2\alpha\alpha_1^{-1}) * \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w}\alpha^{n-1}{\alpha_1}^{m-1}\alpha_2^{w-1}e^{-\alpha D}e^{-\alpha_1 D_1}e^{-\alpha_2 D_2}\,d\alpha\, d\alpha_1 d\alpha_2\,,$$

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right)_J = \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w} * \Big[6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n-1}\,{\alpha_1}^{(m-2)-1}{\alpha_2}^{(w+2)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-2)-1}\,{\alpha_1}^{m-1}{\alpha_2}^{(w+2)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-2)-1}\,{\alpha_1}^{(m-2)-1}{\alpha_2}^{(w+4)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-1)-1}\,{\alpha_1}^{(m-1)-1}{\alpha_2}^{(w+2)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 4\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-1)-1}\,{\alpha_1}^{(m-2)-1}{\alpha_2}^{(w+3)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-2)-1}\,{\alpha_1}^{(m-1)-1}{\alpha_2}^{(w+3)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 4\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+1)-1}\,{\alpha_1}^{(m-2)-1}{\alpha_2}^{(w+1)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n-1)-1}\,{\alpha_1}^{m-1}{\alpha_2}^{(w+1)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n-1}\,{\alpha_1}^{(m-1)-1}{\alpha_2}^{(w+1)-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+2)-1}\,{\alpha_1}^{(m-2)-1}{\alpha_2}^{w-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n-1}\,{\alpha_1}^{m-1}{\alpha_2}^{w-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2 + 2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+1)-1}\,{\alpha_1}^{(m-1)-1}{\alpha_2}^{w-1} * e^{-D\alpha}\,e^{-D_1\alpha_1}\,e^{-D_2\alpha_2}\,d\alpha\, d\alpha_1 d\alpha_2\Big],$$

By solving the integrations which is kernels of gamma dist.

$$= \frac{D^n}{\Gamma n}\frac{D^m}{\Gamma m}\frac{D_2^w}{\Gamma w}\Bigg[6\frac{\Gamma(n)}{D^n}\frac{\Gamma(m-2)}{{D_1}^{m-2}}\frac{\Gamma(w+2)}{{D_2}^{w+2}} + \frac{\Gamma(n-2)}{D^{n-2}}\frac{\Gamma(m)}{{D_1}^m}\frac{\Gamma(w+2)}{{D_2}^{w+2}} + \frac{\Gamma(n-2)}{D^{n-2}}\frac{\Gamma(m-2)}{{D_1}^{m-2}}\frac{\Gamma(w+4)}{{D_2}^{w+4}} + 6\frac{\Gamma(n-1)}{D^{n-1}}\frac{\Gamma(m-1)}{{D_1}^{m-1}}\frac{\Gamma(w+2)}{{D_2}^{w+2}} + 4\frac{\Gamma(n-1)}{D^{n-1}}\frac{\Gamma(m-2)}{{D_1}^{m-2}}\frac{\Gamma(w+3)}{{D_2}^{w+3}} + 2\frac{\Gamma(n-2)}{D^{n-2}}\frac{\Gamma(m-1)}{{D_1}^{m-1}}\frac{\Gamma(w+3)}{{D_2}^{w+3}} + 4\frac{\Gamma(n+1)}{D^{n+1}}\frac{\Gamma(m-2)}{{D_1}^{m-2}}\frac{\Gamma(w+1)}{{D_2}^{w+1}} + 2\frac{\Gamma(n-1)}{D^{n-1}}\frac{\Gamma(m)}{{D_1}^m}\frac{\Gamma(w+1)}{{D_2}^{w+1}} + 6\frac{\Gamma(n)}{D^n}\frac{\Gamma(m-1)}{{D_1}^{m-1}}\frac{\Gamma(w+1)}{{D_2}^{w+1}} + \frac{\Gamma(n+2)}{D^{n+2}}\frac{\Gamma(m-2)}{{D_1}^{m-2}}\frac{\Gamma(w)}{{D_2}^w} + \frac{\Gamma(n)}{D^n}\frac{\Gamma(m)}{{D_1}^m}\frac{\Gamma(w)}{{D_2}^w} + \frac{\Gamma(n+1)}{D^{n+1}}\frac{\Gamma(m-1)}{{D_1}^{m-1}}\frac{\Gamma(w)}{{D_2}^w}\Bigg],$$

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right)_J = \Bigg[1 + 6\frac{(w+1)(w)D_1^2}{(m-1)(m-2)D_2^2} + \frac{(w+1)(w)D^2}{(n-1)(n-2)D_2^2} + \frac{(w+3)(w+2)(w+1)(w)D^2D_1^2}{(n-1)(n-2)(m-1)(m-2)D_2^4} + 6\frac{(w+1)(w)DD_1}{(n-1)(m-1)D_2^2} + 4\frac{(w+2)(w+1)(w)DD_1^2}{(n-1)(m-1)(m-2)D_2^3} + 2\frac{(w+2)(w+1)(w)D^2D_1}{(n-1)(n-2)(m-1)Q_2^3} + 4\frac{(n)(w)D_1^2}{(m-1)(m-2)DD_2} + 2\frac{(w)D}{(n-1)D_2} + 6\frac{(w)D_1}{(m-1)D_2} + \frac{(n+1)(n)D_1^2}{(m-1)(m-2)D^2} + 1 + \frac{(n)D_1}{(m-1)D}\Bigg]. \qquad (23)$$

Substituting equation (22) and equation (23) in equation (21) to get the Bayesian estimation using Non informative Jeffrey's prior based on Quadratic loss function

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Bayesian estimation Using informative prior :

The prior distribution of the parameters $(\alpha, \alpha_1, \alpha_2)$ is gamma distribution with hyper – parameters $(a, a_1, a_2, b, b_1, b_2)$ with pdf's as follows [10] :

$$\Pi(\alpha) = \frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha}. \qquad (24)$$

$$\Pi(\alpha_1) = \frac{b_1^{a_1}}{\Gamma a_1}\alpha_1^{a_1-1}e^{-b_1\alpha_1}. \qquad (25)$$

$$\Pi(\alpha_2) = \frac{b_2^{a_2}}{\Gamma a_2}\alpha_2^{a_2-1}e^{-b_2\alpha_2}. \qquad (26)$$

Then the posterior for $(\alpha, \alpha_1, \alpha_2)$ will be as follows:

Since $\underline{x}, \underline{t}$ *and* $\underline{z}$ *are independent* r.v.'s the posterior distribution for each parameter can be found as:

$$P(\alpha, \alpha_1, \alpha_2|\underline{x}, \underline{t}, \underline{z}) = P(\alpha|\underline{x})\, P(\alpha_1|\underline{t})\, P(\alpha_2|\underline{z}), \qquad (27)$$

$$P(\alpha|\underline{x}) \propto L(\underline{x}|\alpha)\, \Pi(\alpha),$$

where $L(\underline{x}|\alpha)$ and $\Pi(\alpha)$ are defined in equations (8), (24) respectively

$$\propto \alpha^n \prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha} \frac{b^a}{\Gamma a}\alpha^{a-1}e^{-b\alpha},$$

$$= \alpha^{n+a-1}e^{-b\alpha}\prod_{i=1}^{n}\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)^{\alpha},$$

$$= \alpha^{n+a-1}e^{-\alpha\left(-\sum_{i=1}^{n}\ln\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right)+b\right)},$$

Which is the kernel of gamma distribution G $(n + a, Q)$

$$\text{where } Q = -\sum_{i=1}^{n}\ln\left(1 - e^{-\left(\frac{\lambda}{x_i}\right)^2}\right) + b\,, \qquad (28)$$

The complete posterior of $P(\alpha|\underline{x})$ is:

$$P(\alpha|\underline{x}) = \frac{Q^{n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-Q\alpha}. \qquad (29)$$

Similarly, the posterior distributions for $\alpha_1 and \alpha_2$ are:

$$P(\alpha_1|\underline{t}) = \frac{{Q_1}^{m+a_1}}{\Gamma(m+a_1)}{\alpha_1}^{m+a_1-1}e^{-Q_1\alpha_1}. \qquad (30)$$

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Where $Q_1 = -\sum_{j=1}^{m} \ln\left(1 - e^{-\left(\frac{\lambda}{t_j}\right)^2}\right) + b_1 ,$ (31)

$$P(\alpha_2|\underline{z}) = \frac{Q_2{}^{w+a_2}}{\Gamma(w+a_2)}\, \alpha_2{}^{w+a_2-1} e^{-Q_2\alpha_2}. \qquad (32)$$

Where $Q_2 = -\sum_{k=1}^{w} \ln\left(1 - e^{-\left(\frac{\lambda}{z_k}\right)^2}\right) + b_2 ,$ (33)

The joint posterior dist. For $(\alpha, \alpha_1, \alpha_2)$ from Substituting equations (29), (30), (32) in equation (27) is:

$$P(\alpha, \alpha_1, \alpha_2|\underline{x}, \underline{t}, \underline{z}) = \frac{Q^{n+a}}{\Gamma(n+a)} \frac{Q_1{}^{m+a_1}}{\Gamma(m+a_1)} \frac{Q_2{}^{w+a_2}}{\Gamma(w+a_2)} \alpha^{n+a-1}\, \alpha_1{}^{m+a_1-1} \alpha_2{}^{w+a_2-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}. \qquad (34)$$

Bayesian estimation using informative gamma prior based on Quadratic loss function :

From equation (21) the estimated reliability function based on Quadratic loss function is defined as:

$$\hat{R} = \frac{E\left(R^{-1}|\underline{x}, \underline{t}, \underline{z}\right)}{E\left(R^{-2}|\underline{x}, \underline{t}, \underline{z}\right)} , \qquad (35)$$

Where

$$E\left(R^{-1}|\underline{x}, \underline{t}, \underline{z}\right) = \int_0^\infty \int_0^\infty \int_0^\infty (1 + 2\alpha_1^{-1}\alpha_2 + \alpha^{-1}\alpha_2 + \alpha^{-1}\alpha_1^{-1}\alpha_2^2 + \alpha\alpha_1^{-1}) * \frac{Q^{n+a}}{\Gamma(n+a)} \frac{Q_1{}^{m+a_1}}{\Gamma(m+a_1)} \frac{Q_2{}^{w+a_2}}{\Gamma(w+a_2)} \alpha^{n+a-1}\, \alpha_1{}^{m+a_1-1} \alpha_2{}^{w+a_2-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2} d\alpha\, d\alpha_1 d\alpha_2 ,$$

$$E\left(R^{-1}|\underline{x}, \underline{t}, \underline{z}\right) = \frac{Q^{n+a}}{\Gamma(n+a)} \frac{Q_1{}^{m+a_1}}{\Gamma(m+a_1)} \frac{Q_2{}^{w+a_2}}{\Gamma(w+a_2)} * \Big[\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{n+a-1}\, \alpha_1{}^{m+a_1-1} \alpha_2{}^{w+a_2-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$2\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{n+a-1}\, \alpha_1{}^{(m+a_1-1)-1} \alpha_2{}^{(w+a_2+1)-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+a-1)-1}\, \alpha_1{}^{m+a_1-1} \alpha_2{}^{(w+a_2+1)-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+a-1)-1}\, \alpha_1{}^{(m+a_1-1)-1} \alpha_2{}^{(w+a_2+2)-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty \int_0^\infty \int_0^\infty \alpha^{(n+a+1)-1}\, \alpha_1{}^{(m+a_1-1)-1} \alpha_2{}^{w+a_2-1} * e^{-Q\alpha}\, e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\, d\alpha\, d\alpha_1 d\alpha_2\Big] ,$$

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By solving the integrations which is kernels of gamma dist.

$$= \frac{Q^{n+a}}{\Gamma(n+a)}\frac{{Q_1}^{m+a_1}}{\Gamma(m+a_1)}\frac{{Q_2}^{w+a_2}}{\Gamma(w+a_2)}\left[\frac{\Gamma(n+a)}{Q^{n+a}}\frac{\Gamma(m+a_1)}{{Q_1}^{m+a_1}}\frac{\Gamma(w+a_2)}{{Q_2}^{w+a_2}} + 2\,\frac{\Gamma(n+a)}{Q^{n+a}}\frac{\Gamma(m+a_1-1)}{{Q_1}^{m+a_1-1}}\frac{\Gamma(w+a_2+1)}{{Q_2}^{w+a_2+1}} + \frac{\Gamma(n+a-1)}{Q^{n+a-1}}\frac{\Gamma(m+a_1)}{{Q_1}^{m+a_1}}\frac{\Gamma(w+a_2+1)}{{Q_2}^{w+a_2+1}} + \frac{\Gamma(n+a-1)}{Q^{n+a-1}}\frac{\Gamma(m+a_1-1)}{{Q_1}^{m+a_1-1}}\frac{\Gamma(w+a_2+2)}{{Q_2}^{w+a_2+2}} + \frac{\Gamma(n+a+1)}{Q^{n+a+1}}\frac{\Gamma(m+a_1-1)}{{Q_1}^{m+a_1-1}}\frac{\Gamma(w+a_2)}{{Q_2}^{w+a_2}}\right],$$

$$E\left(R^{-1}\middle|\underline{x},\underline{t},\underline{z}\right) = 1 + 2\frac{(w+a_2)Q_1}{(m+a_1-1)Q_2} + \frac{(w+a_2)Q}{(n+a-1)Q_2} + \frac{(w+a_2+1)(w+a_2)QQ_1}{(n+a-1)(m+a_1-1)Q_2^2} + \frac{(n+a)Q_1}{(m+a_1-1)Q}.$$
(36)

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right) = \int_0^\infty\int_0^\infty\int_0^\infty R^{-2}\ P\left(\alpha,\alpha_1,\alpha_2\middle|\underline{x},\underline{t},\underline{z}\right) d\alpha\, d\alpha_1 d\alpha_2\,,$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty (6\alpha_1^{-2}\ \alpha_2^2 + \alpha^{-2}\alpha_2^2 + \alpha^{-2}\alpha_1^{-2}\alpha_2^4 + 6\alpha^{-1}\alpha_1^{-1}\alpha_2^2 + 4\alpha^{-1}\alpha_1^{-2}\alpha_2^3 + 2\alpha^{-2}\alpha_1^{-1}\alpha_2^3 + 4\alpha\alpha_1^{-2}\alpha_2 + 2\alpha^{-1}\alpha_2 + 6\alpha_1^{-1}\alpha_2 + \alpha^2\alpha_1^{-2} + 1 + 2\alpha\alpha_1^{-1}) * \frac{Q^{n+a}}{\Gamma(n+a)}\frac{{Q_1}^{m+a_1}}{\Gamma(m+a_1)}\frac{{Q_2}^{w+a_2}}{\Gamma(w+a_2)}\alpha^{n+a-1}\ {\alpha_1}^{m+a_1-1}{\alpha_2}^{w+a_2-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2} d\alpha\, d\alpha_1 d\alpha_2\,,$$

Where Q, Q_1, Q_2 are defined in equations (28), (31), (33) respectively

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right) = \frac{Q^{n+a}}{\Gamma(n+a)}\frac{{Q_1}^{m+a_1}}{\Gamma(m+a_1)}\frac{{Q_2}^{w+a_2}}{\Gamma(w+a_2)} * \Big[6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n+a-1}\ {\alpha_1}^{(m+a_1-2)-1}{\alpha_2}^{(w+a_2+2)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-2)-1}\ {\alpha_1}^{m+a_1-1}{\alpha_2}^{(w+a_2+2)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-2)-1}\ {\alpha_1}^{(m+a_1-2)-1}{\alpha_2}^{(w+a_2+4)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-1)-1}\ {\alpha_1}^{(m+a_1-1)-1}{\alpha_2}^{(w+a_2+2)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$4\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-1)-1}\ {\alpha_1}^{(m+a_1-2)-1}{\alpha_2}^{(w+a_2+3)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-2)-1}\ {\alpha_1}^{(m+a_1-1)-1}{\alpha_2}^{(w+a_2+3)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$4\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a+1)-1}\ {\alpha_1}^{(m+a_1-2)-1}{\alpha_2}^{(w+a_2+1)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$
$$2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a-1)-1}\ {\alpha_1}^{m+a_1-1}{\alpha_2}^{(w+a_2+1)-1} * e^{-Q\alpha}\ e^{-Q_1\alpha_1}\ e^{-Q_2\alpha_2}\ d\alpha\, d\alpha_1 d\alpha_2 +$$

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$$6\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n+a-1}\,\alpha_1^{(m+a_1-1)-1}\alpha_2^{(w+a_2+1)-1} * e^{-Q\alpha}\; e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2 +$$
$$\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a+2)-1}\,\alpha_1^{(m+a_1-2)-1}\alpha_2^{w+a_2-1} * e^{-Q\alpha}\; e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2 + \int_0^\infty\int_0^\infty\int_0^\infty \alpha^{n+a-1}\,\alpha_1^{m+a_1-1}\alpha_2^{w+a_2-1} * e^{-Q\alpha}\; e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2 +$$
$$2\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{(n+a+1)-1}\,\alpha_1^{(m+a_1-1)-1}\alpha_2^{w+a_2-1} * e^{-Q\alpha}\; e^{-Q_1\alpha_1}\, e^{-Q_2\alpha_2}\; d\alpha\, d\alpha_1 d\alpha_2 \Big],$$

By solving the integrations which is kernels of gamma dist.

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right) = \frac{Q^{n+a}}{\Gamma(n+a)}\frac{Q_1^{m+a_1}}{\Gamma(m+a_1)}\frac{Q_2^{w+a_2}}{\Gamma(w+a_2)} * \Big[\, 6\frac{\Gamma(n+a)}{Q^{n+a}}\frac{\Gamma(m+a_1-2)}{Q_1^{m+a_1-2}}\frac{\Gamma(w+a_2+2)}{Q_2^{w+a_2+2}} +$$
$$\frac{\Gamma(n+a-2)}{Q^{n+a-2}}\frac{\Gamma(m+a_1)}{Q_1^{m+a_1}}\frac{\Gamma(w+a_2+2)}{Q_2^{w+a_2+2}} + \frac{\Gamma(n+a-2)}{Q^{n+a-2}}\frac{\Gamma(m+a_1-2)}{Q_1^{m+a_1-2}}\frac{\Gamma(w+a_2+4)}{Q_2^{w+a_2+4}} + 6\frac{\Gamma(n+a-1)}{Q^{n+a-1}}\frac{\Gamma(m+a_1-1)}{Q_1^{m+a_1-1}}\frac{\Gamma(w+a_2+2)}{Q_2^{w+a_2+2}} +$$
$$4\frac{\Gamma(n+a-1)}{Q^{n+a-1}}\frac{\Gamma(m+a_1-2)}{Q_1^{m+a_1-2}}\frac{\Gamma(w+a_2+3)}{Q_2^{w+a_2+3}} + 2\frac{\Gamma(n+a-2)}{Q^{n+a-2}}\frac{\Gamma(m+a_1-1)}{Q_1^{m+a_1-1}}\frac{\Gamma(w+a_2+3)}{Q_2^{w+a_2+3}} +$$
$$4\frac{\Gamma(n+a+1)}{Q^{n+a+1}}\frac{\Gamma(m+a_1-2)}{Q_1^{m+a_1-2}}\frac{\Gamma(w+a_2+1)}{Q_2^{w+a_2+1}} + 2\frac{\Gamma(n+a-1)}{Q^{n+a-1}}\frac{\Gamma(m+a_1)}{Q_1^{m+a_1}}\frac{\Gamma(w+a_2+1)}{Q_2^{w+a_2+1}} +$$
$$6\frac{\Gamma(n+a)}{Q^{n+a}}\frac{\Gamma(m+a_1-1)}{Q_1^{m+a_1-1}}\frac{\Gamma(w+a_2+1)}{Q_2^{w+a_2+1}} + \frac{\Gamma(n+a+2)}{Q^{n+a+2}}\frac{\Gamma(m+a_1-2)}{Q_1^{m+a_1-2}}\frac{\Gamma(w+a_2)}{Q_2^{w+a_2}} + \frac{\Gamma(n+a)}{Q^{n+a}}\frac{\Gamma(m+a_1)}{Q_1^{m+a_1}}\frac{\Gamma(w+a_2)}{Q_2^{w+a_2}} +$$
$$\frac{\Gamma(n+a+1)}{Q^{n+a+1}}\frac{\Gamma(m+a_1-1)}{Q_1^{m+a_1-1}}\frac{\Gamma(w+a_2)}{Q_2^{w+a_2}} \Big],$$

$$E\left(R^{-2}\middle|\underline{x},\underline{t},\underline{z}\right) = \Big[6\frac{(w+a_2+1)(w+a_2)Q_1^2}{(m+a_1-1)(m+a_1-2)Q_2^2} + \frac{(w+a_2+1)(w+a_2)Q^2}{(n+a-1)(n+a-2)Q_2^2} +$$
$$\frac{(w+a_2+3)(w+a_2+2)(w+a_2+1)(w+a_2)Q^2Q_1^2}{(n+a-1)(n+a-2)(m+a_1-1)(m+a_1-2)Q_2^4} + 6\frac{(w+a_2+1)(w+a_2)QQ_1}{(n+a-1)(m+a_1-1)Q_2^2} +$$
$$4\frac{(w+a_2+2)(w+a_2+1)(w+a_2)QQ_1^2}{(n+a-1)(m+a_1-1)(m+a_1-2)Q_2^3} + 2\frac{(w+a_2+2)(w+a_2+1)(w+a_2)Q^2Q_1}{(n+a-1)(n+a-2)(m+a_1-1)Q_2^3} + 4\frac{(n+a)(w+a_2)Q_1^2}{(m+a_1-1)(m+a_1-2)QQ_2} +$$
$$2\frac{(w+a_2)Q}{(n+a-1)Q_2} + 6\frac{(w+a_2)Q_1}{(m+a_1-1)Q_2} + \frac{(n+a+1)(n+a)Q_1^2}{(m+a_1-1)(m+a_1-2)Q^2} + 1 + \frac{(n+a)Q_1}{(m+a_1-1)Q}\Big]. \qquad (37)$$

Substituting equations (36) and (37) in equation (35) to get the Bayesian estimation using informative gamma prior based on Quadratic loss function (Q.L.F).

V. Simulation study

In this part, a simulation study is conducted to identify the most effective estimator for the stress-strength reliability (S-SR.) of Exponentiated Inverse Rayleigh distribution three estimators had been found for (S-SR.) which is (Maximum likelihood estimator $\hat{R}(mle)$, Bayesian estimator using Non informative Jeffrey's prior based on Quadratic loss function ($\hat{R}_{BJ}$) , Bayesian estimation using informative gamma prior based on Quadratic loss function ($\hat{R}_{BC}$)), and An evaluation of the mean square error (MSE) was conducted for the estimators

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using different sample sizes (25,50,150) when ($\lambda = 0.5$, $\alpha = 0.5$, $\alpha_1 = 0.92$, $\alpha_2 = 1.3$) and for gamma priors ($a = 4$, $a_1 = 3.8$, $a_2 = 5.39$, $b = 2.2$, $b_1 = 2.7$, $b_2 = 2.09$) for 1000 replicates and the The simulation study was carried out using (R Studio) to evaluate the performance of the stress-strength reliability (S-SR) estimator, following the steps outlined below:

A. Generate random values for $\underline{x}, \underline{t}$ *and* $\underline{z}$ by the inverse function according to:

$x = \lambda/[-\ln(1-(1-u)^{\frac{1}{\alpha}})]^{1/2}$ where u is generated from the uniform distribution.

B. Calculate the mean of the estimators by $\frac{\sum_{i=1}^{n} \hat{R}_i}{length(\hat{R}_i)}$

C. Evaluate the mean square error (MSE) for the estimators $MSE = \frac{\sum_{i=1}^{n}(\hat{R}_i - R)^2}{length(\hat{R}_i)}$,

And the estimator with lowest Mean square error (MSE) considered the best estimator under that size.

TABLE I. SIMULATION RESULTS WHEN $\lambda = 0.5$

Samples Size (n,m,w)		MLE	$\hat{R}_{BJ}$	$\hat{R}_{BC}$
(25,25,25)	Mean	0.09626097	0.08317858	0.08287311
	MSE	0.0008124976	0.000825472	0.0007002272
(50,50,50)	Mean	0.09496247	0.08850988	0.08780993
	MSE	0.0003917041	0.0004025999	0.0003713655
(150,150,150)	Mean	0.09435836	0.09247435	0.09207806
	MSE	0.0001283046	0.0001308556	0.0001271602
(25,25,50)	Mean	0.09557486	0.08540304	0.088966
	MSE	0.0005338496	0.0005542385	0.0004666725
(25,25,150)	Mean	0.09567359	0.08750723	0.09442819
	MSE	0.0003528695	0.0003634629	0.0003139656

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(25,50,50)	Mean	0.0946362	0.08656192	0.08864342
	MSE	0.0004434233	0.0004728347	0.0004118427
(25,150,50)	Mean	0.09478747	0.08807479	0.08908944
	MSE	0.0003841617	0.0004101916	0.0003599019
(25,150,150)	Mean	0.09500985	0.09037138	0.09468963
	MSE	0.0002212373	0.0002356505	0.0002135279
(50,25,25)	Mean	0.09596393	0.084448	0.08150327
	MSE	0.0007274677	0.0007379786	0.0006581071
(50,25,50)	Mean	0.09438281	0.08588607	0.08681341
	MSE	0.0004733879	0.0005008539	0.0004391084
(50,25,150)	Mean	0.09570129	0.08919527	0.09335837
	MSE	0.0003179193	0.0003182683	0.0002799644
(50,150,50)	Mean	0.09460014	0.08951676	0.08776759
	MSE	0.0003263493	0.0003396176	0.0003197875
(50,150,150)	Mean	0.09467149	0.09167879	0.09322179
	MSE	0.0001597035	0.0001648208	0.0001558222
(150,25,25)	Mean	0.09524273	0.084786	0.08006148
	MSE	0.000663068	0.0006830251	0.0006401942
(150,25,50)	Mean	0.09450405	0.08702942	0.08603887
	MSE	0.0004120928	0.0004323197	0.0003916446
(150,25,150)	Mean	0.09541274	0.08998477	0.09224112
	MSE	0.0002703758	0.0002724102	0.0002356616

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(150,50,50)	Mean	0.09477342	0.08938146	0.08678581
	MSE	0.0003501634	0.0003571205	0.000342188
(150,50,150)	Mean	0.09526524	0.09197389	0.09262799
	MSE	0.0001734437	0.0001723454	0.0001615088

VI._ Conclusion and Discussion

The result of the experiment in table (1) shows that:

1- $R = 0.09395425$

2- Bayesian estimation using informative gamma prior based on Quadratic loss function is the best estimator because it has smallest MSE.

The reliability decrease (life of the component is affected and reduced with High stress (the component subject to low stress is more reliable than the component subject to high stress for the same component).

As a future studies, stress-strength reliability (S-SR.) function P(T<X<Z) under cencored data could be presented and discussed.

VII._References

[1] S. Nadarajah and S. Kotz, "The exponentiated type distributions," Acta Applicandae Mathematicae, vol. 92, 2, pp. 97–111, 2006..

[2] S. Kotz, Y. Lumelskii, and M. Pensky, "The stress- strength model and its generalizations, Theory and Applications", World Scientific Publishing Co.Pte.Ltd., 2003.

[3] N. Singh, "On the estimation of Pr(T<X<Z)", Communication in Statistics Theory & Methods, 9, pp. 1551-1561, 1980.

[4] Hassan, Amal S.; Elsayed, A. Elsherpieny; Rania, M. Shalaby. "On the Estimation of for Weibull Distribution in the Presence of k Outliers". *International Journal of Engineering Research and Applications*, 2013, 3.6: 1728-1734.

[5] Patwray, A. N.; Sriwastav, G. L.; Hazarika, J. "Inference of R= P (X< Y<Z) for n-Standby System: A Monte-Carlo Simulation Approach". *J. Math*, 2016, 12: 18-22.S

[6] Mutair, A., and Karam, N. S. "Stress-Strength Reliability for P (T< X<Z) using The New Weibull-Pareto Distribution". Al-Qadisiyah Journal Of Pure Science, (2021), 26(2), 39-51.

[7] B. Abdulhameed, A. N. Salman, and B. A. Kalaf," On the Estimation of $(Y_1 < X < Y_2)$ in Cased Inverted Kumaraswamy Distribution", Iraqi Journal of Science Vol. 61, No. 4, pp. 845-853, , 2020.

ISSN: 2392- 5418 **e-ISSN:** 2520- 7423 **DOI:**10.59799/SQZH5356

[8] Y. Aliyu , and A. Yahaya ," Bayesian estimation of the shape parameter of generalized Rayleigh distribution under non-informative prior", International Journal of Advanced Statistics and Probability, 4 (1) ,pp. 1-10, 2016.

[9] N. H. Al-Noor, H. A. A. Al-Ameer," Some Estimation Methods for the Shape Parameter and Reliability Function of Burr Type XII Distribution / Comparison Study", Mathematical Theory and Modeling, Vol.4, No.7, pp. 63-77, 2014.

G. Srinivasa Rao, S. Mbwambo, and A. Pak, "Estimation of multicomponent stress-strength reliability from exponentiated inverse Rayleigh distribution", Journal of Statistics & Management Systems, DOI: 10.1080/09720510.2020.1761094, Aug 2020.